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Author(s): Mary Ann Huntley, Chris L. Rasmussen, Roberto S. Villarubi, Jaruwan Sangtong, James T. Fey
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# Effects of Standards-Based Mathematics Education: A Study of the Core-Plus Mathematics Project Algebra and Functions Strand 

Mary Ann Huntley<br>National Center for Improving Science Education, Washington, DC<br>Chris L. Rasmussen, Purdue University, Calumet<br>Roberto S. Villarubi, University of Maryland<br>Jaruwan Sangtong, University of Maryland<br>James T. Fey, University of Maryland


#### Abstract

To test the vision of Standards-based mathematics education, we conducted a comparative study of the effects of the Core-Plus Mathematics Project (CPMP) curriculum and more conventional curricula on growth of student understanding, skill, and problem-solving ability in algebra. Results indicate that the CPMP curriculum is more effective than conventional curricula in developing student ability to solve algebraic problems when those problems are presented in realistic contexts and when students are allowed to use graphing calculators. Conventional curricula are more effective than the CPMP curriculum in developing student skills in manipulation of symbolic expressions in algebra when those expressions are presented free of application context and when students are not allowed to use graphing calculators.


Key Words: Algebra; Conceptual knowledge; Functions; Program/project assessment; Reform in mathematics education; Representations, modeling

Recent recommendations by major mathematics education professional organizations, such as the National Council of Teachers of Mathematics (NCTM) and the Mathematical Sciences Education Board (MSEB), call for fundamental changes in secondary school mathematics curricula, instruction, and assessment (NCTM, 1989, 1991, 1995b; National Research Council, 1989). The proposed changes include design of curricula with a common core of broadly useful mathematics for all students, emphasis on student-centered instruction that engages students in exploration of mathematical facts and principles through collaborative work on

[^0]authentic problems, and assessment of student learning through a variety of strategies that are embedded in regular classroom activity.

In secondary school mathematics the proposed changes in curriculum, teaching, and assessment have special implications for the treatment of algebra. The concepts, principles, and techniques of algebra are important tools for describing and reasoning about patterns in all branches of mathematics. Thus algebra has been at the heart of secondary school mathematics for many years, and high achievement in algebra has long been the hallmark of preparedness for advanced mathematical and scientific studies. Traditional curricula treat concepts and skills of algebra in two separate yearlong college-preparatory courses (usually separated by a year of deductive geometry), but authors of reform documents envision a curriculum that features integrated strands of algebra and functions, geometry and trigonometry, statistics and probability, and discrete mathematics for all students. As reported by Kieran (1992), traditional algebra instruction features teacher explanation and student practice of routine symbol-manipulation skills, not student exploration of authentic problems that are infused with algebraic ideas. Traditional assessments of algebraic knowledge also emphasize questions that call for routine performance of symbol-manipulation procedures, not application of those skills in significant problem solving.

Many new approaches to secondary school mathematics in general and algebra in particular are based on the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). Those Standards reflect the best judgment of many professionals who applied their knowledge of research on mathematics teaching and learning, their insights into implications of emerging technological aids for mathematical calculation, and their practical classroom experiences to formulate proposals for reform. To test the complex array of proposals for change, researchers needed to construct and implement full curricula that model the Standards principles. The Core-Plus Mathematics Project (CPMP) is one such Standards-based approach to secondary school mathematics. The purpose of this study was to test the vision of Standards-based mathematics education, using the CPMP treatment of algebra and functions as a specific case. In particular, we sought evidence comparing the effects of the CPMP curriculum and more conventional curricula on growth of student understanding, skill, and problem-solving ability in algebra.

In this article, we first provide a background for the study by describing the CorePlus Mathematics Project. We then outline our overall research and curriculumdesign perspective on algebra, representations, and mathematical modeling. The research questions, our methods of data collection and analysis, and the results of our analysis are then presented. Finally, we conclude by discussing the significance of this study for Standards-based reform in secondary school mathematics, especially in the treatment of algebra.

## THE CPMP PROGRAM

The Core-Plus Mathematics Project was funded by the National Science Foundation in 1992 to construct a 3-year integrated mathematics curriculum for
all students plus a 4th-year course continuing the preparation of students for college mathematics. The resulting curriculum is based on the theme that mathematics is a tool for making sense of the world around us. The curriculum materials pose questions about real-life and mathematical contexts and investigations that will lead students to important mathematical understandings and skills. Each year of the CPMP curriculum features topics in algebra and functions, geometry and trigonometry, statistics and probability, and discrete mathematics. The four major content strands are unified by common themes like data, symbolic reasoning, variability, representation, shape, and change; by common topics like graphical representations, curve fitting, and matrices; and by habits of mind like visual thinking, recursive thinking, reasoning with multiple representations, and providing convincing arguments (Hirsch, Coxford, Fey, \& Schoen, 1995).

The 3-year core curriculum is organized in 21 connected units, each designed for 4 to 6 weeks of classroom study. Each unit is comprised of several multiday lessons in which major ideas are developed through investigations of applied problems. Core topics are intended to be accessible and engaging to all students, and lessons are designed to promote small-group cooperative learning. Use of graphing calculators is a prominent feature of the curriculum-to enable all students to develop versatile ways of dealing with realistic situations and to remove barriers that have, in the past, prevented large numbers of students from continuing their study of significant mathematics.

After pilot testing of the first CPMP course materials was completed, national field tests of the curriculum began with Course 1 in 1994-95, Course 2 in 1995-96, and Course 3 in 1996-97. As is fairly typical in curriculum innovations of this sort, few schools were able to cover all 21 units in the field test. However, the units in the algebra and functions strand were almost always covered in full.

## Perspective on Algebra

One of the striking features of current proposals for reform of school mathematics is the great difference between traditional and innovative curricula in objectives and presentation of algebra. Traditional curricula have generally been focused on training students in a variety of procedural skills for manipulating polynomial and rational expressions in order to solve equations. This traditional approach to school algebra relies almost exclusively on written symbolic forms as the carriers of algebraic ideas.

The NCTM Standards and numerous other recent proposals for reform of school algebra reflect different views of the central ideas in algebra and of the methods by which those ideas should be taught and executed. When those new views of algebra are translated into working school mathematics programs, they can take a variety of forms and approaches (Bednarz, Kieran, \& Lee, 1996; Davis, 1993; Kaput, 1997; Katz, 1997; NCTM, 1995a; van Reeuwijk, 1995). Three central tenets underlie CPMP's perspective on algebra. First, the primary role of algebra at the school level is to provide effective models of numerical patterns and quantitative relations-in pure mathematics and in the many applications of mathematics
in which numerical data are important. Second, the key concepts in algebraic modeling are variables and functions. Third, the use of graphing calculators and computers makes the focus on modeling and functions attractive and accessible for students across a broad range of interests, aptitudes, and prior achievement. Use of calculating tools also offers students a variety of powerful new learning and problem-solving strategies and diminishes the need for students to acquire a high degree of skill in symbol manipulation (Fey, 1989).

The CPMP development of algebra presents each idea and method in at least three linked representations-graphic, numeric, and symbolic. CPMP students, often using technology, learn about a variety of functions through numerical and graphical explorations. In addition to performing symbolic manipulations, students systematically search graphs and tables of function values to solve traditional algebra problems like equations and inequalities as well as previously inaccessible problems involving optimization and rates of change. The CPMP approach emphasizes laying a strong conceptual foundation for use of symbolic forms. Reasoning skills that guide formal symbol manipulation are developed at points later in the curriculum than in more traditional mathematics programs, and units in which algebra is emphasized are integrated with and connected to units in the other content strands.

There are seven units in CPMP Courses 1-3 with a primary emphasis on algebra and functions: Course 1 includes "Patterns of Change" (unit 2), "Linear Models" (unit 3), and "Exponential Models" (unit 6); Course 2 includes "Power Models" (unit 4); and Course 3 includes "Multivariable Models" (unit 1), "Symbol Sense and Algebraic Reasoning" (unit 3), and "Families of Functions" (unit 6). Algebra concepts and skills are applied and extended in other units of the curriculum, such as in "Matrix Models" and "Patterns of Location, Shape, and Size" (Course 2) and in "Discrete Models of Change" (Course 3). In all, primary attention to algebra and functions takes up slightly more than one third of the 3-year core curriculum.

## Perspective on Representations and Modeling

Underlying much of the CPMP approach to algebra is the perspective taken on representations and modeling. As mentioned previously, CPMP students approach mathematical ideas through investigations of applied problems using a variety of linked representations. For example, in one activity called Modeling a Bungee Apparatus, students collect data on the relationship between length of a stretched bungee cord and the amount of weight attached to the end of that cord. As part of the activity, students represent certain aspects of the problem situation with tables and graphs, and they generally discover a fairly linear relationship between weight and stretch length (though it should be emphasized that students are not necessarily representing what we understand to be a linear function but rather are depicting aspects of the problem that are meaningful to them). Students then go on to explore different contexts in which linear relationships exist between variables of interest. As a consequence of these activities and the ensuing classroom discussions, students are expected to build rich and varied connections among aspects of the
concept of a linear function. Similar sequences of activities in a variety of contexts focus on exponential functions, quadratic functions, and other fundamental families of functions.

The perspective on multiple representations just described is compatible with the viewpoint elaborated by Kaput (1993) and P. W. Thompson (1994). For example, Thompson posited that
our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead by our making connections among representational activities. (p. 39)

In other words, a sense of connection among representations (such as tables, graphs, and expressions) for the concept of a linear function, for example, is enabled through a variety of experiences with representational activities and applied-problem settings. In this manner CPMP uses and relies on representations and representational activities to further students' mathematical development.

A further point of clarification is necessary regarding the CPMP view of mathematical modeling and this research project. In general, through mathematical modeling one seeks to represent a complex situation in terms of quantitative or spatial and visual relationships to learn more about the situation under investigation (Putnam, Lampert, \& Peterson, 1990). As shown in Figure 1, the process of mathematical modeling can be depicted as a cyclic process involving mathematical formulation of an applied-problem situation, mathematical activity within and among representations, and interpretation of the results in terms of the original situation. Interpretations may result in further and revised mathematical formulation and more mathematical activity with representations.

If one is concerned with only the results of students' written work, then it is reasonable to think about formulation as moving from an applied problem to a math-


Figure 1. Mathematical modeling schema.
ematical model and interpretation as moving from the mathematical model back to the applied situation. However, from the perspective of our research, this directionality is inadequate to describe the mental activities of students. In Figure 1, the arrows between the boxes labeled Applied-Problem Situation and Mathematical Activity are intentionally bidirectional. The applied-problem situation and the mathematical realm of tables, graphs, and symbols are dialectically related. Thus, although the written solution of a problem may suggest linear reasoning from an applied situation to the mathematical model and vice versa, students' mental activities may very well involve complex interplay between the applied situation and the mathematical representation. Thus, it is in this respect that the arrows in Figure 1 point in both directions between the applied-problem situation and activity with the mathematical model.

## Summary of Framework and Goals of the Study

The Core-Plus Mathematics Project is one of the major national efforts to construct, implement, and evaluate a high school mathematics program exemplifying principles and practices recommended in recent proposals for reform. The seven algebra and functions units form one of the major strands of the Core-Plus Mathematics Program, and they approach traditional mathematical topics in new ways: emphasizing mathematical modeling; using graphing calculators to support multiple representations of algebraic ideas; learning through collaborative work on authentic problems; integrating algebra with topics in geometry, statistics, probability, and discrete mathematics; organizing topics in a concept-then-skills-thenabstraction order; and reducing attention to formal symbol-manipulation procedures.

Our broad purpose in this research was to compare the effects of such an algebra experience for students to the effects of more conventional high school mathematics curricula. Consistent with our perspective on the nature of algebra in school mathematics, we looked in particular at students' abilities to formulate quantitative problems in algebraic form, to carry out the algebraic thinking and calculations required to answer questions, and to interpret the results of algebraic reasoning and calculations in authentic problem contexts.

## RESEARCH DESIGN AND METHODOLOGY

Assessing and comparing student learning from different curricula are complex problems requiring hard choices among data-collection and analysis options. It makes sense to assess knowledge of students with comparable mathematical aptitude and interest prior to curricular treatments, but in the world of real schools such ideal samples are not easy to construct. It makes some sense to assess students on mathematical topics they have had equal opportunities to learn, but not when the goals of programs being compared are different in many significant respects. Timing of assessments and incentives to get good student effort on test instruments
are not simple issues, and one has the further choice between studying a few students through intensive individual interviews or studying a large number of students through less insightful performance measures.

When we considered these dilemmas, our thinking was informed by designs of several major studies in which researchers attempted to assess effects of curricula and teaching. In the 1960s the School Mathematics Study Group (SMSG) conducted an extensive longitudinal study of "new math" curriculum innovations (Begle, 1979). Designers of the SMSG National Longitudinal Study of Mathematical Abilities (NLSMA) used a broad battery of assessment instruments to construct profiles of achievement for various kinds of curricula instead of using single-score comparisons on a core set of objectives. The mathematics and science studies administered by the National Assessment of Educational Progress (NAEP) and the International Association for the Evaluation of Educational Achievement (IEA) are often reported as single score "horse-race" mathematics competitions, but in those studies student achievement is actually assessed on a broad range of content. Thus it is possible to use NAEP and IEA data to construct profiles of achievement across many specific mathematical topics. Furthermore, both NAEP and IEA studies use item-sampling procedures that provide rich achievement data without extensive testing time for individual students. The IEA researchers have paid particular attention to the relationships among intended, implemented, and achieved curricula (McKnight et al., 1987; Schmidt, McKnight, \& Raizen, 1997).

Our basic path through the maze of design options was to use a battery of paper-and-pencil instruments to assess the understanding, skill, and problem-solving ability of CPMP and control students ending their 3rd year of high school mathematics. Through teacher interviews we developed profiles of the intended and implemented curricula at each research site, and we used a variety of strategies to assure comparability of students being tested. The following sections describe the research procedures and our rationale for choosing them.

## Population

Early in the Spring 1997 semester we sent letters to lead teachers in 36 high schools that are national field-test sites for the Core-Plus Mathematics Project, inviting their Course 3 CPMP teachers to participate in a comparative study of algebraic reasoning. We chose the end of Course 3 for testing because at this point in the integrated CPMP curriculum the core algebra and functions units are completed. We expected that students in the control classes would be drawn primarily from advanced algebra classes-again nearing the end of their high school algebra experience.

Participation in the study was based on three criteria: (a) implementation of the CPMP program with something approximating recommended conditions (heterogeneous grouping, covering the intended curriculum units, and using technology and cooperative learning); (b) identification within the school or a neighboring school of classes using traditional curricula with students of comparable ability;
and (c) willingness to devote two classroom sessions to testing. Six U.S. schools accepted our invitation: two in the Southeast, two in the Midwest, one in the South, and one in the Northwest. At each site there were two CPMP teachers and one, two, or three control teachers. The number of students varied from approximately 90 to 180 per site.

As might be expected, establishing comparability of students in the CPMP and control groups was not easy. In four of the sites we were able to obtain standardized mathematics test scores from eighth grade for most students involved in the study. At one of these sites, these data showed us that the control and CPMP groups were of comparable ability on entry to high school; at the other three of these sites we used blocking techniques to construct samples of comparable ability-CPMP students were matched with control students who had comparable mathematics achievement or aptitude scores in Grade 8. At one more site students had been randomly assigned to CPMP and control treatments on entry to Grade 9. At the remaining test site we were unable to gain release of eighth-grade test data, but we received repeated assurances that the tested student groups were of comparable ability and just happened to get into different curricular tracks at the start of high school.

## Development of Instruments

Two types of data were collected for the study. First, we developed interview protocols to obtain qualitative information from each of the participating CPMP and control teachers. To describe variability inherent in different implementations of the curricula, we designed the interview protocols to gather information about teachers' instructional practices and curriculum coverage. We asked each teacher about additions to or deletions from the intended curriculum, typical classroom instructional practices, use of calculators, reactions to the CPMP curriculum, and assessment practices.

Second, we developed various instruments to assess students' understanding, skill, and problem-solving abilities in algebra and functions. The schematic diagram in Figure 1 identifies three main components of effective algebraic thinking: (a) using algebraic ideas and techniques to mathematize quantitative problem situations, (b) using algebraic principles and procedures like solution of equations and inequalities to produce results beyond the information given in the original situation, and (c) interpreting results of mathematical reasoning and calculations in the problematic situation.

Students who have effective command of algebra and functions can execute the complete process outlined in Figure 1 with skill and understanding. They can use that knowledge to solve significant problems, to discover and confirm important algebraic principles, and to inform decision making in situations that depend on quantitative factors. Therefore, we assessed student performance on comprehensive problems involving all phases of the framework presented in Figure 1. At the same time, each component of algebraic problem-solving and reasoning activity
requires a variety of constituent understandings and skills. Thus we assessed these components of algebraic thinking separately. Finally, because many high-stakes college-admission and placement tests still require skill in algebraic symbol manipulation without the aid of technology, we assessed performance on those skills in questions devoid of meaningful problem contexts. In summary, we developed three algebra assessments, each with several parallel forms of roughly the same difficulty. This research plan allowed us to get information about many aspects of algebra knowledge.

The first assessment (Part 1) emphasized the type of contextualized problem solving that is quite typical of CPMP units and other reform curricula. There were four parallel forms of Part 1, each with four superproblems (a problem setting with five to seven questions asked about that situation). Some of those problems were based on items from the American Mathematical Association of Two-Year Colleges (Cohen, 1995) and items by Gordon, Gordon, Fusaro, Siegel, and Tucker (1995). Other problems were written by our research team. A portion of a superproblem similar to those in Part 1 is given in Figure 2.

The second assessment (Part 2) emphasized context-free symbolic manipulations that call for transformation of algebraic expressions and solutions of equations and

## Problem 2: $\quad$ The Long-Distance Airliner

Several commercial airlines have non-stop flights from Los Angeles, California, all the way to Sydney, Australia. It is a trip of 7500 miles and can take as long as 18 hours, most of the time out over the Pacific Ocean. Therefore, estimating flight time and fuel requirements is very important. One airline uses a formula to predict flight time $T$ in hours from wind speed $W$ in miles per hour.

$$
T=\frac{7500}{500+W}
$$

Answer questions 2.1-2.3 that occur in use of the formula to make safe flight plans. Remember to show your work. If you use a calculator, explain how.

Question 2.1: $\quad$ Find $T$ when $W=-50$ and explain what the result tells about trip flight plans.

Question 2.2: Find the wind speed that will give a flight time of 14 hours.
Question 2.3: Explain the information that will be given by solving the inequality

$$
T=\frac{7500}{500+W}>14 \text { for } W
$$

Figure 2. Sample superproblem from Assessment Part 1.
systems. There were two parallel forms of this assessment, each with six multiplechoice questions and eight constructed-response questions. These questions were adapted from items on released ACT examinations and items that commonly appear on college placement tests. Two questions typical of those in the Part 2 assessment are given in Figure 3.
2. Which of the following expressions is equivalent to $\frac{125+x}{25}$ ?
(a) $5 x$
(b) $5+x$
(c) $100+x$
(d) $\frac{126 x}{25}$
(e) $5+\frac{x}{25}$
9. Solve the system of equations $\left\{\begin{array}{l}-2 x+3 y=8 \\ x-y=2\end{array}\right.$

Work Space: Answer:

Figure 3. Sample items from Assessment Part 2.

The third assessment (Part 3) required collaborative work on open-ended contextual problems. There were three parallel forms of this assessment, each consisting of one question encompassing all three phases of activity outlined in Figure 1. These problems were written by the research team. One of the three problems is given in Figure 4.

## Data Collection

All data were collected during April and May of 1997 by project staff (including three of the authors and two CPMP staff at other sites) who visited each site to assist with that process. Data were derived from interviews and assessments. The semistructured interviews with CPMP and control teachers were conducted individually. All were audiotaped and took at most one hour.

The assessments of student algebra knowledge were administered over the course of 2 days. All class sessions were approximately 50 minutes in duration. Teachers were asked to inform their students about the purpose and general form of the tests prior to the first day of testing. On the first day of testing, students were given 50 minutes to complete Part 1. The several different but parallel forms were randomly distributed to students in each class. Students were to work individually, and use of scientific or graphing calculators was permitted. On the second day of testing,

## Selling by Telephone

Many companies sell their products through long-distance telephone calls to customers. For example, the $C D$ Club sells music compact discs all across the country from its headquarters in New York. Sales calls made by the $C D$ Club last an average of about 4 minutes apiece. The club has bids from three possible providers for their long-distance telephone service:

1) Apple Communications will charge $\$ 0.35$ for placing each call and then $\$ 0.15$ per minute of time used in the call.
2) Bell Telephone makes a fixed charge of $\$ 260$ per week for access to its long-distance lines, but charges only $\$ 0.10$ per minute of time used by the calls.
3) Capital Long Distance Services will make a fixed charge of $\$ 600$ per week for unlimited use of its long-distance lines.

Question: For the CD Club the problem is to choose the long-distance service that is least expensive for their business. What advice would you give about the phone company to choose and why?

Conclusions and Reasoning:

Figure 4. Sample problem from Assessment Part 3.
students were given 20 minutes to complete Part 2 of the assessment. Again they were to work individually, but this time without use of calculators. The two parallel forms of Part 2 were distributed randomly to students in each class. After the completed Part 2 assessments were collected, students were paired and each pair was given 20 minutes to complete one randomly assigned form of the Part 3 assessment. Use of scientific or graphing calculators was permitted on Part 3; graph paper and rulers were provided; and each pair of students submitted one answer paper.

## Data Analysis

The author team conducted all data analyses. In every case at least two of us looked at each piece of interview data or student work to confirm interpretation and scoring; discrepancies were resolved by a third person.

Interview data. Analysis of the interview data shed light on the variability with which the Core-Plus Mathematics and control curricula are implemented at each site. For teachers in the national field test of CPMP, each year of teaching the material was a first for them, and the students tested were a subset of those selected for CPMP classes in ninth grade, 3 years before the testing in our study. Different partic-
ipating schools chose to implement the CPMP curriculum with different student populations-in some cases with students above average in aptitude and prior achievement, in some cases with students of below-average aptitude and prior achievement, and in only one case with students representing the full spectrum of mathematical aptitude and prior achievement. Thus the implemented curriculum varied greatly from site to site.

By the time of our testing, teachers and students at five of the six schools had worked on nearly all parts of the three CPMP algebra units in Course 3 as well as on the four algebra units in Courses 1 and 2. At the remaining site (Site 1), the CPMP classes had completed Courses 1 and 2 but had only begun work on the Course 3 algebraic-reasoning unit at the time of testing. Both CPMP and control classes at that site consisted largely of students well below average in prior mathematics aptitude and attainment. In all CPMP field-test sites, students had routine access to graphing calculators for all class work, and in most sites they were allowed to take the calculators out of school for homework. Field-test teachers reported frequent use of collaborative small-group activities in their instruction, though most of the teachers were still evolving instructional strategies for managing that kind of classroom work.

Interviews with control teachers also revealed considerable variability across sites with respect to the curriculum and instruction their students had experienced in their 3 years of high school mathematics. Although several current teachers of control students reported using advanced algebra textbooks from commercial publishers, one used a discrete-mathematics-with-applications textbook, one used a textbook that emphasized applications of mathematics, and one used a textbook that focused on the use of mathematics in business settings. Most of those books were fairly traditional in content and presentation, but some reflected influence of Standards recommendations. Some teachers reported limited use of scientific or graphing calculators and cooperative learning, but the common pattern was fairly traditional high school mathematics instruction.

We collected data regarding the placement of students into CPMP and traditional curricula at the various sites as well as factors that influenced this decision. At one site students were randomly assigned to CPMP and control classes at the beginning of Grade 9. At other sites students were given the option of entering CPMP or a traditional-curriculum track in Grade 9. At some sites the CPMP curriculum was used with students of generally weaker aptitude and interest because of a desire to upgrade the content of curricula for such students and a general feeling that there would be little risk in trying the new program with students who had not succeeded in conventional curricula.

Student-achievement data. We developed rubrics to score student papers. The rubrics were validated by having someone outside of the project evaluate the general rubrics and item-specific criteria-first without reference to student work and then with some samples of student work to clarify the meaning of the scoring criteria for each item. After coding test papers so graders would not know the name,
site, or treatment group for any students, four of the five authors scored the papers. The rubrics were based on the following general principles:

- On items in Parts 1 and 2, full credit was given if the answer was correct, regardless of work shown. Because students in more traditional mathematics curricula tend to have less experience communicating their work and reasoning, we adopted this principle to give fair credit for correct work.
- In all three parts of the assessment, partial credit was given for responses showing evidence (i.e., supporting work) of progress toward a correct solution. If there was no supporting work shown, partial credit was given in some cases in which an incorrect answer was obviously obtained from errant mental calculation or calculator use, but the error indicated correct thinking. No partial credit was given for the six multiple-choice test items on the two forms of the Part 2 assessment.
- Minor errors (e.g., arithmetic errors) received a small deduction if the task was multistep but received a large deduction if the task was not very involved.
- Major errors (e.g., algebraic errors such as combining unlike powers of $x$ or omitting one root of a quadratic equation) received large deductions.

Each item on each part of the assessment was graded independently by two people. Student scores on each question were then entered into an SPSS database (Version 6.1.3) for further analyses.

Imposing something like a legitimate experimental design and statistical analysis on something as complex as a national field test of a 3-year curriculum is a real challenge. We considered various ways to analyze the data-one being to report on six separate cases. In the end we reasoned that over the course of 3 years the individual CPMP and control students would have had quite varied experiences, all driven by either a reform or fairly traditional curriculum. We did not believe that analysis with teachers or classrooms as the unit of analysis was appropriate, and we became convinced that our various strategies for assuring comparable CPMP and control-student pools at the various sites were in fact providing groups with comparable entering mathematical aptitude and achievement.

In deciding on specific statistical tests to run on the data, we faced another dilemma. With many possible research questions to consider and data from many kinds of assessment items, we were wary of running too many statistical tests and getting spurious indicators of differences between treatments. Thus we decided to reserve our statistical power for main-effect tests as much as possible. Our decision was to run statistical tests on broad questions and then to present breakdowns of the data in the spirit of data-snooping-looking for interesting patterns in specific-item statistics underlying the main effects. In some cases we pooled results from different forms and parts of the tests, so we were uncertain that even generous interpretation of assumptions for statistical tests could be satisfied. Thus in the results that we present in the next section, we give test statistics and $p$-values for the main analyses and further tables that show raw performance data on interesting specific items or families of items.

## RESULTS

The structure of the various assessment forms and the database of item scores for each student allowed us to conduct a variety of global and specific analyses in which performance of CPMP and control students was compared and contrasted.

## Main Effects

The instruments we used to assess student algebraic understanding, technical skill, and problem-solving ability included more than 100 questions covering a broad range of algebraic ideas, relationships, and techniques. But, in some sense, those specific tasks were designed to help us answer two main questions about the effects of the Core-Plus Mathematics curriculum:

- Is the CPMP program more effective than a conventional curriculum in developing student ability to solve algebraic problems when those problems are presented in realistic contexts and when the students are allowed to use technological tools like graphing calculators?
- Are conventional curricula more effective than the CPMP program in developing student skills in manipulation of symbolic expressions in algebra when those expressions are presented free of application context and when students are not allowed to use tools like graphing calculators?
As explained in the methodology section, the specific test questions were presented in three separate parts. The four forms of Part 1 included questions that asked students to formulate and use algebraic models to answer various questions about relationships among variables. Use of graphing calculators was allowed on Part 1. The two forms of Part 2 included questions about equivalence of algebraic expressions and solution of equations and inequalities with no application contexts and no access to calculators. The three forms of Part 3 each included one fairly open problem in an applied context. Students generally worked in pairs on one such problem and used graphing or scientific calculators, graph paper, and rulers as desired.

Algebra in context. Development of algebraic ideas through modeling of quantitative relationships in contextual problems is emphasized in the algebra strand of the Core-Plus Mathematics curriculum. CPMP students are also encouraged to make extensive use of graphing calculators as tools for exploring algebraic ideas and solving algebraic problems. Thus, one would expect CPMP students to do better than control students in Parts 1 and 3 of the algebra assessment. As indicated by results in Tables 1 and 2, that was the case in our testing. On questions that required specific algebraic skills and problem-solving strategies like translating problem conditions into symbolic expressions, solving equations, and interpreting results (Part 1) and on problems that required integration of those specific skills for work on a more complex modeling task (Part 3), CPMP students outperformed students of comparable mathematical aptitude who had experienced a more traditional mathematical curriculum.

Table 1
Performance on Applied Algebra Problems With Use of Calculators-Part 1 (All Sites Combined)

| Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 273 | 34.1 | 14.8 |
| CPMP | 320 | 42.6 | 21.3 |

Note. $t_{570}=-5.69, p<.001$.

Table 2
Performance on Open-Ended Applied Algebra Problems With Use of Calculators-Part 3 (All Sites Combined)

| Treatment | $n$ | $M(0-4)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 191 | 1.07 | 1.20 |
| CPMP | 184 | 1.43 | 1.35 |

Note. $t_{364}=-2.77, p<.01$.

The differences between treatments were consistent across forms and items in Part 1. Each form consisted of 4 superproblems, clusters of questions based on a single applied-problem context. On 15 of the 16 superproblems, CPMP students outperformed control students. CPMP students also outperformed control students on two of the three forms of Part 3, with CPMP and control scores on the third form essentially the same.

Symbol manipulation. Although use of algebra in contextual problem solving is emphasized in the Core-Plus Mathematics curriculum, each unit that develops a major algebraic idea includes sections in which the key structures and techniques are identified and abstracted. Furthermore, a major unit in Course 3 (11th grade) is titled "Symbol Sense and Algebraic Reasoning." The focus of that unit is rules for symbol manipulation and their basis in the ordered-field properties of the rational and real number systems. Thus CPMP students at the end of Course 3 are expected to have some skill in doing symbolic algebra independent of application context and without use of graphing-calculator technology. Their skills in this area were expected to be weaker than those of students in a conventional curriculum that includes 2 full years of largely symbolic algebra, and results from our testing confirmed that expectation.

On questions that required pure algebraic symbol manipulation-testing equivalence of expressions and solving equations and inequalities-students in the control classes outperformed CPMP students of comparable abilities. Results in Table 3 show a difference of 9.4 percentage points between the group means on the symbol-manipulation-skill items. The difference between treatments was consistent across forms and items within the forms. On only 2 of the 28 items did the CPMP group significantly outperform the control group; the control-group mean was significantly greater on 15 of the 28 items.

Table 3
Performance on Algebraic Symbol Manipulation Without Use of Calculators-Part 2 (All Sites Combined)

| Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 265 | 38.4 | 16.2 |
| CPMP | 312 | 29.0 | 18.4 |

Note. $t_{575}=6.50, p<.001$.

Variability by site. As one would expect, the aggregate algebra-achievement data mask considerable variability across the settings in which CPMP national-fieldtest schools are located. Testing for our study was done at six very different school sites with quite different populations of students at the various sites. To understand the variability of implementation that might be expected from a curriculum innovation like CPMP, we analyzed data from the three parts of the algebra assessment at each site.

Table 4
Comparison of CPMP and Control Groups by Site

| Site | Treatment | $n$ | Part $1 M(0-100)$ |  | Part $2 M(0-100)$ |  | Part $3 M(0-4)$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | M | SD | M | SD | M | SD |
| 1 | Control | 24 | 31.7 | 9.6 | 33.9 | 13.4 | 0.58 | 1.12 |
| 1 | CPMP | 25 | 35.5 | 14.0 | 20.4 | 12.1 | 0.96 | 1.12 |
| 2 | Control | 48 | 26.0 | 10.8 | 32.0 | 13.2 | 0.40 | 0.65 |
| 2 | CPMP | 34 | 49.4 | 18.4 | 29.9 | 18.4 | 1.50 | 1.37 |
| 3 | Control | 28 | 36.7 | 13.1 | 49.8 | 15.8 | 1.64 | 1.50 |
| 3 | CPMP | 27 | 25.2 | 14.1 | 15.3 | 10.8 | 0.50 | 1.10 |
| 4 | Control | 92 | 41.9 | 14.8 | 38.8 | 14.9 | 1.54 | 1.24 |
| 4 | CPMP | 77 | 47.7 | 21.8 | 38.3 | 20.1 | 1.72 | 1.49 |
| 5 | Control | 40 | 29.4 | 14.7 | 40.1 | 19.9 | 1.16 | 1.12 |
| 5 | CPMP | 54 | 38.3 | 22.2 | 24.1 | 15.7 | 1.65 | 1.30 |
| 6 | Control | 36 | 30.5 | 15.1 | 37.8 | 15.5 | 0.79 | 1.03 |
| 6 | CPMP | 94 | 45.6 | 21.2 | 29.8 | 17.5 | 1.71 | 1.22 |

Note. Numbers of students taking each part of the assessment varied somewhat around the average $n$ for each site.

Results in Table 4 show that the CPMP curriculum was implemented with considerable variability of effect in the different sites, with the overall theme that CPMP students tended to do better on algebraic tasks embedded in appliedproblem contexts when graphing calculators were available, whereas controlgroup students did better on traditional symbol-manipulation tasks. Of course, in interpreting this variability, we must consider as well the facts that CPMP students and their teachers were the first group in their schools using the new curriculum and that the curriculum itself has subsequently been revised and developed further to reflect experiences of the pilot and national-field-test experiences.

## Specific Effects of CPMP and Control Curricula

The overall summaries of student performance on applied algebraic problem solving and symbol manipulation show fairly consistent and not surprising differences between effects of CPMP and traditional curricula. However, a number of detailed analyses illuminate those effects and suggest ways that curriculum developers can modify materials and teachers can modify implementation of those materials to improve student learning. On both the applied-problem-solving and the symbol-manipulation assessments there is considerable room for improvement in student performance. In the sections that follow we examine several aspects of algebra achievement in more detail.

Formulation and interpretation of mathematical models. In 49 specific questions of the 16 superproblems in the Part 1 assessment, students were asked either to formulate an algebraic model or to interpret results from use of a given algebraic model. These questions highlight the kind of mathematical activity that is characteristic of the reflexive relationship between problematic situations and conventional mathematical formulations. Considering the overall results from Part 1, one might expect that CPMP students would perform better than the control students on these types of problems. Results given in Table 5 confirm this expectation. CPMP students outperformed control students on all but 5 of the 49 questions.

Table 5
Performance on Problems That Involve Setting Up Algebraic Models or Interpreting Results of Algebraic Calculations

| Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 273 | 30.5 | 16.7 |
| CPMP | 319 | 39.3 | 22.4 |

Note. $t_{580}=-5.44, p<.0001$.

Looking separately at results from the questions that required students to move from a context to a conventional mathematical representation and vice versa seems reasonable. However, we urge the reader to be cautious in drawing inferences about student thinking from such results. We consider conventional mathematical formulations and realistic situations to be dialectically related. Thus, although a student's written work on a particular problem might appear to reflect reasoning from an applied-problem situation directly to a mathematical representation (or vice versa), the student's mental activities might not have followed such a simple path.

With that caveat, we next consider two subsets of the problems in the previous collection-one subset that calls for mathematical formulation of given contextual information and another that calls for interpretation of conventional mathematical representations. Of the 49 items in this collection, 24 of the problems involve setting up algebraic models and 25 items involve interpretation of algebraic models.

Figure 5a shows a typical problem resulting in mathematical formulation, and 5b shows a typical problem resulting in interpretation.

5(a) The Watchdog Security Service provides home and business security systems with installation charge of $\$ 150$ and a charge of $\$ 5$ per week to monitor the system. Write a formula that gives cost $C$ of the service as a function of time $t$ in weeks that the service is needed.

5(b) If you want to see things that are far away, it's natural to climb to some high spot like the top of an observation tower or tall building. The maximum distance you can see across level ground is a function of your height above ground with rule $d(x)=1.219 \sqrt{x}$ where $x$ is in feet above ground and viewing distance $d(x)$ is in miles. What would you learn from calculating $d(100)$ ? You do not need to do the calculations.

Figure 5. Items requiring formulation and interpretation of a model.

Analyses for each of the two subsets gave results that are similar to those shown in Table 5. CPMP students performed better than control students on items that involved formulating algebraic models of quantitative relationships and on those items that involved interpreting the results of algebraic calculations. Means comparisons for these two subsets of questions are shown in Table 6. The results in Table 6 show that the differences between CPMP students and control students are consistent across problem type, with CPMP students outperforming control students on both formulation questions and interpretation questions.

Table 6
Performance on Questions Involving Formulation of Mathematical Models and Interpretation of Algebraic Calculations

|  | Formulation $M(0-100)$ |  |  | Interpretation $M(0-100)$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Treatment | $M$ | $S D$ |  | $M$ | $S D$ |
| Control $(n=273)$ | 23.4 | 19.6 |  | 38.6 | 19.8 |
| CPMP $(n=320)$ | 32.8 | 28.4 |  | 46.4 | 22.9 |

Algebraic calculation and reasoning. Formulating an algebraic function rule, equation, or inequality is only the first step in effective quantitative problem solving. To draw meaningful conclusions from given inormation, one invariably needs to perform algebraic calculations as well-to evaluate expressions, to solve equations and inequalities, and to transform expressions into useful equivalent
forms. In traditional curricula, students do the required calculations by following procedural rules for manipulation of symbolic expressions. In curricula that make use of numeric, graphic, and symbolic calculating tools, students have several more options available for answering such questions. Furthermore, in Standards-based curricula that make heavy use of real-world contexts for teaching algebraic ideas, students are encouraged to use contextual metaphors as guides to thinking about algebraic tasks. Many of the specific questions in our Part 1 and Part 2 assessments yielded data for comparison of performance by CPMP and control students on alge-braic-calculation tasks.

Student performance on algebraic calculations was measured by 18 items on Part 1. Those calculations were embedded in problem contexts, and students had full access to calculators (graphing or scientific) for their work. As shown in Table 7, the mean score of CPMP students (57.4\%) was higher, but not significantly higher, than that of control students (53.9\%).

Table 7
Performance on Algebraic Calculations in Context With Access to Calculators

| Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 273 | 53.9 | 28.5 |
| CPMP | 320 | 57.4 | 32.1 |

Note. $t_{589}=-1.39, p<.164$.

The 18 algebraic-calculation items of Part 1 included questions requiring substituting specific values of variables in expressions, solving equations, and solving inequalities. The relative performance of CPMP and control students in those separate categories is shown in Table 8. In addition to recognizing the advantages of contextual metaphors to support reasoning about algebraic expressions, one might guess that student access to and familiarity with use of graphing calculators would be helpful on these traditionally difficult problems.

Table 8
Performance on Evaluating Expressions, Solving Equations, and Solving Inequalities in Context With Access to Calculators M (0-100)

| Treatment | Evaluating expressions |  | Solving equations |  | Solving inequalities |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | M | SD | M | SD | $M$ | SD |
| Control | $\begin{gathered} 71.5 \\ (n=273) \end{gathered}$ | 34.8 | $\begin{gathered} 42.1 \\ (n=204) \end{gathered}$ | 45.8 | $\begin{gathered} 21.3 \\ (n=211) \end{gathered}$ | 32.1 |
| CPMP | $\begin{gathered} 68.7 \\ (n=320) \\ \hline \end{gathered}$ | 35.8 | $\begin{gathered} 48.4 \\ (n=241) \\ \hline \end{gathered}$ | 46.7 | $\begin{gathered} 30.6 \\ (n=244) \\ \hline \end{gathered}$ | 36.0 |

Looking even more closely at the student performance on algebraic-calculation items reveals some interesting and useful insights into curricular effects. For example, CPMP and control students demonstrated similar abilities to accurately
evaluate linear and quadratic expressions. However, on two items requiring substitution of numerical values into algebraic fractions, performance of control students was 30 percentage points higher than that of CPMP students. One problem called for evaluation of $T=7500 /(500+W)$ for $W=-50$. A number of students answered -35 , suggesting mistaken or thoughtless use of the calculator keystroke sequence $7500 \div 500-50$.

CPMP students generally solved equations more successfully than control students. For example, Figure 6 shows one such question from a Part 1 superproblem that requires analysis equivalent to solving a linear equation. The mean score of CPMP students on this item was $61.0 \%$ whereas the mean score of control students was only $44.5 \%$.

The Turtle Mountain Springs Company made plans for growth in their share of the water business. They predicted that annual income from sale of their bottled water $B$ and filters $F$ would change over time according to the following formulas. Time $t$ is in years since 1990 and income is in millions of dollars per year.

> Bottled Water Income: $B=20+5 t$
> Filtering Devices Income: $F=28+3 t$

Question: When does the Turtle Mountain Springs Company expect the two water products to give the same annual income?

Figure 6. One question from Superproblem C2.

Results on items asking for solution of inequalities revealed a similar pattern of stronger CPMP student performance on items stated in a problem context. For example, a superproblem on one form of Part 1 gave the equation $D=0.1 s^{2}+0.7 s$ for predicting stopping distance $D$ of a large truck moving at $s$ miles per hour. One specific question in this problem was "Over what range of speeds can the trucker drive and still be able to stop in at most 400 feet?" For this item, students needed to make calculations equivalent to solving the inequality $0.1 s^{2}+0.7 s \leq 400$. Performance of CPMP students on this item ( $46.8 \%$ ) was better than that of control students $(21.2 \%)$. In contrast, results given below show that when neither context clues nor calculators were available for such a task, control students outperformed CPMP students.

Because of the difficulty in interpreting students' written explanations for their answers, we were unable to determine whether CPMP students performed better on such items because they had a wider repertoire of problem-solving strategies, including calculator use, or whether they had acquired better ability to recognize
the mathematical task embedded in a contextual problem. The results suggest strengths of newer approaches to algebra. However, the general level of performance by CPMP students leaves room for improvement, indicating that new curricula might need to provide better instruction in use of multiple algebraic strategies.

Twenty-two of the 28 items on the Part 2 tests were designed to assess student skill with algebraic calculations when no context clues or calculator assistance was available. (The other 6 problems involved mathematization, representational fluency, and finding the slope of a line given in standard form.) Consistent with results from the overall analysis of Part 2 items, presented earlier in Table 3, control students outperformed CPMP students in these conditions. The results appear in Table 9.

Table 9
Performance on Algebraic Calculations Without Context or Calculator Access

| Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: |
| Control | 265 | 41.9 | 16.9 |
| CPMP | 312 | 30.7 | 20.1 |

Note. $t_{575}=7.25, p<.001$.

We were not surprised that CPMP students, whose program does not focus on symbolic manipulation by paper and pencil, attained lower scores than traditional students whose program consisted of symbolic manipulation almost exclusively. Table 10 shows that the scores of control students were higher than the scores of CPMP students in every subcategory: evaluating expressions, testing equivalence of expressions, solving equations, and solving inequalities.

Table 10
Performance on Evaluating Expressions, Testing Equivalence, Solving Equations, and Solving Inequalities Without Context or Calculator Access M (0-100)

| Treatment | Evaluating expressions |  | Testing equivalence |  | Solving equations |  | Solving inequalities |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | M | SD | M | SD | M | SD | M | $S D$ |
| Control | $\begin{gathered} 73.9 \\ (n=138) \end{gathered}$ | 34.3 | $\begin{gathered} 48.0 \\ (n=265) \end{gathered}$ | 26.9 | $\begin{gathered} 35.1 \\ (n=265) \end{gathered}$ | 20.9 | $\begin{gathered} 53.1 \\ (n=265) \end{gathered}$ | 43.3 |
| CPMP | $\begin{gathered} 61.9 \\ (n=162) \end{gathered}$ | 37.7 | $\begin{gathered} 36.6 \\ (n=312) \end{gathered}$ | 26.8 | $\begin{gathered} 24.9 \\ (n=312) \end{gathered}$ | 22.4 | $\begin{gathered} 35.7 \\ (n=312) \end{gathered}$ | 42.9 |

Once again, details of performance on specific items reveal some interesting insights. For example, on 12 items requiring solution of equations, the greatest difference in mean scores between control and CPMP students involved the linear equation $3 x+4=5 x-2$. The mean scores of control and CPMP students were $84.8 \%$ and $49.4 \%$, respectively. Students would have used this type of equation to answer the question about income for Turtle Mountain Springs Water in Part 1 (see

Figure 6), but on that item, presented as a question in context, CPMP students outperformed control students. This result illustrates once again the general proposition that CPMP students perform better than control students when setting up models and solving algebraic problems presented in meaningful contexts while having access to calculators, but CPMP students do not perform as well on formal symbol-manipulation tasks without access to context cues or calculators. The pattern of results highlights the choices that one makes in selection of curricula and teaching methods and suggests modifications of reform and traditional algebra programs that would be needed to reach additional objectives.

Representational fluency. Traditional approaches to school algebra focus almost exclusively on use of symbolic expressions to represent operations and relationships involving quantitative variables. One of the principal arguments for reform of this traditional approach to algebra is the conjecture that the integrated numeric, graphic, and symbolic tools of modern calculators and computers provide powerful new ways of learning and doing mathematics. The NCTM Standards authors argued that students who become fluent in applying and translating among those multiple representations "will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics" (NCTM, 1989, p. 146).

The CPMP curriculum is consistent with these recommendations for the use of multiple representational activities to further students' mathematical development. With the aid of graphing calculators, each family of algebraic expressions and relationships is presented in at least three linked representations-numeric, graphic, and symbolic. Hence, many of the items in Parts 1 and 2 of our assessment were designed to shed light on how well CPMP students and control students were able to translate among the different representations.

In 14 items, spanning Parts 1 and 2, students were asked to move among symbolic, tabular, and graphical representations of information. One such problem is given in Figure 7. Because of the contextualized nature of most problems on Part 1 of the assessment and the decontextualized nature of problems on Part 2, three analyses were performed on those item scores. First, we analyzed overall performance on questions requiring representational fluency. Second, we analyzed performance on the 8 fluency problems that were set in meaningful contexts. Third, we analyzed performance on the 6 fluency items that were not embedded in problem contexts. Table 11 shows the results of those analyses. The numbers of students whose scores were analyzed are somewhat lower than in earlier analyses because we included data only from students whose randomly assigned test forms included items related to representational fluency.

As shown in Table 11, CPMP students generally outperformed control students on problems involving movement among symbolic, tabular, and graphic representations. As expected, the performance for CPMP students was much better on tasks set in context and tasks for which calculator use was allowed than on tasks devoid of meaningful context and tasks for which calculator use was not allowed. Again, this result reflects the relative emphases of the two approaches to algebra.

Write an equation relating $x$ and $y$ that will give the pairs of numbers in this table.

| $x$ | -1 | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 3 | 5 | 7 | 9 | 11 |

Figure 7. A problem involving representational fluency.

Table 11
Performance on Representational Fluency Items (Symbols, Tables, Graphs)

| Problem type | Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: | :---: |
| Overall (14 problems) | Control | 187 | 19.5 | 18.2 |
| Problems with context ${ }^{\mathrm{a}}$ (8 problems) | CPMP | 224 | 27.3 | 21.2 |
|  | Control | 207 | 22.4 | 31.3 |
| Problems without context $^{\mathrm{b}}$ (6 problems) | CPMP | 240 | 3.7 | 37.7 |
|  | Control | 131 | 18.9 | 20.8 |
|  | CPMP | 156 | 22.5 | 22.8 |

${ }^{\text {a }}$ Calculators available to all students. ${ }^{\mathrm{b}}$ Calculators available to students for 3 of the 6 problems.

Another interesting question about representational fluency concerns students' abilities to translate information from one specific representational type to another. We examined the 14 problems requiring such fluency by problem type: tabular to symbolic (3 problems), graphical to symbolic (4 problems), symbolic to graphical ( 6 problems), and tabular and graphical to symbolic ( 1 problem). In the last category, students were given information in both tabular and graphic formats and were asked to write a symbolic expression for the pattern of data pairs.

Table 12 reports mean performance of CPMP and control students for each type of representational translation task. The largest disparity between control and CPMP students occurred for tabular-to-symbolic translation tasks. Control students had particular difficulty with such problems. Both groups of students fared best on problems that required translation from a symbolic to a graphical representation.

In summary, as might be expected, students following the CPMP algebra program, which emphasizes multiple representations of algebraic ideas, were better able to deal with mathematical tasks requiring representational fluency. However, with mean scores on all items falling below $33 \%$, there is clearly considerable room for improvement in this aspect of mathematical reasoning.

Conceptual and procedural knowledge. One of the most important research problems raised by current proposals to reform school mathematics is understanding the connections between development of conceptual and procedural knowledge in

Table 12
Performance on Movement Between Specific Pairs of Representations

| Problem type | Treatment | $n$ | $M(0-100)$ | $S D$ |
| :--- | :---: | :---: | :---: | :---: |
| Table to symbol (3 problems) | Control | 207 | 14.3 | 27.9 |
| Graph to symbol (4 problems) | CPMP | 241 | 23.8 | 36.4 |
|  | Control | 131 | 20.3 | 30.6 |
| Symbol to graph (6 problems) | CPMP | 156 | 23.2 | 28.9 |
|  | Control | 89 | 24.3 | 18.6 |
| Table and graph to symbol (1 problem) | CPMP | 107 | 33.4 | 20.5 |
|  | Control | 187 | 22.0 | 25.1 |
|  | CPMP | 225 | 29.8 | 29.7 |

algebra (Hiebert, 1986). Training students in formal symbol manipulation has always been justified by the argument that without such skill students would be helpless in algebraic problem solving. Because of the numeric, graphic, and symbolic tools provided by today's calculators and computers, many mathematics educators (Davis, 1993; Fey, 1989; Kaput, 1997) have suggested that students might need to know only how to plan and interpret algebraic calculations, not to be proficient in the procedures themselves. Contemporary researchers on mathematics learning have suggested that students might be able to develop conceptual understanding of important algebraic ideas without prior acquisition of proficiency in procedural skills (Hiebert \& Carpenter, 1992). But many mathematicians and teachers are convinced that ability to perform symbol manipulations is an essential correlate of (if not prerequisite to) conceptual understanding and problem solving with algebraic expressions and equations. The various assessments used in our study allowed us to gain insight into this crucial issue from a variety of perspectives.

The general reversal of performance by CPMP and control groups on Parts 1 and 2 of our algebra assessment raises questions about "skills before problem solving" claims. To complement this information, we then looked at the correlations of scores for individual students on the two types of algebra tests. For students in control classes, the correlation was 26 and for students in the CPMP classes the correlation was .35 . Although both correlations are significantly different from 0 , they show that performance on one test explains a very low fraction of the variance in scores on the other. Not only do these low correlations provide further evidence that skill in algebraic symbol manipulation is not a prerequisite for problem solving, they also suggest that many students who are not particularly successful in a traditional calculation-oriented course (A. G. Thompson, Philipp, Thompson, \& Boyd, 1994) might be empowered by alternative approaches to algebra that emphasize meaningful problems and utilize computational technology.

In the design of the Part 1 test instruments for our study, we structured the various forms to give insight into two specific questions about conceptual and procedural knowledge: (a) How will the ability of CPMP students to plan algebraic manipulations compare to their ability to do those calculations accurately? and (b) How will the ability of CPMP students to do algebraic symbol manipulations compare
to their ability to interpret results of those calculations in problem settings? We intended to present some students with contextual problems requiring algebraic calculations and to give other students questions in identical contexts but requiring only planning or interpretation of results from symbolic manipulations. Limitations in the numbers of participants and testing time restricted the number of plan-do item pairs. However, in 11 pairs of items we were able to examine the relative difficulty of doing and interpreting algebraic manipulations.

For example, one test form included questions about economics of a motorcycle business:

The builders of a new American Eagle motorcycle plan to set a fixed price for the cycle-no rebates or bargaining allowed. The problem is finding the right price to charge. They estimate that their operating costs $c$ will be related to the number $n$ of cycles that are made and sold: $c=800 n+10,032,000$.

Students who took one form of the test were asked, "What will the operating cost be if 5,000 cycles are made and sold?" Students who took another form of the test were asked to explain briefly the information given by the statement "If $n=5,000$, then $c=14,032,000$." They were told not to check the calculations that might be involved in producing that information. When we looked at these and other similar pairs of related items, we were particularly interested to see whether ability to carry out the procedural calculations of algebraic problem solving was either a necessary or a sufficient condition for ability to interpret results correctly. The other 10 item pairs used to study this issue involved evaluation of functions (or interpretation of given function values); solution of equations and inequalities with linear, quadratic, rational, and square root functions (or interpretation of given solutions); and construction of graphs to satisfy given conditions (or interpretation of given graphs).

The performance data on the 22 items are extremely varied, and there is no obvious way to summarize the pairwise comparisons. Table 13 shows mean percentages of the possible maximum score obtained by CPMP and control students on the 11 item pairs. It shows a pattern that might be expected. On the calculation items involved in the do-interpret pairs of our testing, performance of the CPMP and control groups was about the same, but CPMP students generally did better on items that called for interpretation of calculated results. The gap between ability to do and ability to interpret algebraic calculations was greater for control students, who presumably had much less practice with the interpretive aspects of problem solving.

For both CPMP and control students, doing algebraic calculations was easier than writing interpretations of the results. There were only two exceptions to this pattern, both involving graphs. In one case, some students were asked to sketch graphs that showed quantities increasing at constant, increasing, and decreasing rates, whereas their counterparts were simply asked to match given graphs with those patterns of change. In the other case, some students were asked to sketch the graph of an inequality in two variables, and their counterparts were asked to interpret points in regions of a given inequality graph.

Table 13
Percentages of Possible Points on Questions Involving "Doing" and "Interpreting" Algebraic Operations

| Problem type | Task | CPMP |  | Control |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | M | $S D$ | M | SD |
| Solve equations and inequalities | Interpret | $21.7(n=237)$ | 35.3 | 11.6 ( $n=200$ ) | 20.5 |
|  | Calculate | 21.3 ( $n=235$ ) | 31.5 | 18.9 ( $n=199$ ) | 29.9 |
| Function values | Interpret | 43.7 ( $n=165$ ) | 41.8 | 29.0 ( $n=145$ ) | 34.8 |
|  | Calculate | $71.4(n=165)$ | 37.8 | $79.9(n=145)$ | 33.0 |
| Graphs | Interpret | $94.5(n=82)$ | 17.6 | $93.8(n=72)$ | 18.6 |
|  | Calculate | 58.0 ( $n=83$ ) | 37.1 | $44.7(n=73)$ | 35.6 |
| Total | Interpret | 32.6 ( $n=320$ ) | 38.6 | $18.4(n=273)$ | 27.6 |
|  | Calculate | $40.4(n=317)$ | 39.4 | $38.5(n=271)$ | 39.6 |

Taken as a whole, and combined with earlier comparisons of performance on problems in context and without context, the data on pairs of items with similar algebraic content but different tasks (do or interpret) give some support to the notion that learning how to interpret results of algebraic calculations is not highly dependent on the ability to perform the calculations themselves. However, even a curriculum in which greater emphasis is placed on mathematical modeling, including interpretation of algebraic calculations in real-world contexts, will not routinely produce students who have mastered that ability.

## DISCUSSION

The broad purpose of this study was to test the vision of Standards-based mathematics education, using the CPMP treatment of algebra and functions as a specific case. Of course, the NCTM Standards $(1989,1991,1995 b)$ are a complex set of interrelated proposals for reform in content, teaching, and assessment of school mathematics $\mathrm{K}-12$, so conducting a single study that will decisively test the broad vision described in the three Standards volumes is nearly impossible. At the same time, no single Standards recommendation can be tested without creating an environment in which most of the interacting proposals have been implemented. Nonetheless, the study described in preceding sections of this report sheds light on a number of general and specific effects of the Standards at the high school level.

## Intended, Implemented, and Achieved Curriculum

The most consistent finding of our algebra assessments is perhaps obviousstudents learn more about topics that are emphasized in their mathematics classes and less about topics that are not emphasized. The content of curriculum text materials and classroom coverage of those materials make a difference. This finding echoes earlier work of NLSMA (Begle, 1979) and a broad finding of IEA assessments like Second International Mathematics Study (McKnight et al., 1987) and Third International Mathematics and Science Study (Schmidt et al., 1997).

Recent reform proposals like the NCTM Standards (1989) have recommended increased attention to certain aspects of algebra and reduced attention to others. Despite the facts that the CPMP curriculum allows more room for practice on items like those found on Parts 1 and 3 of the assessment and the control curricula allow more room for practice on items like those found on Part 2 of the assessment, students' scores were generally low on all types of algebra items. However, in comparing student learning from reform and more traditional algebra courses, we found that students whose algebra instruction emphasizes use of functions and graphing technology to solve authentic quantitative problems become more adept at solving such problems than students whose work includes less applied-problem solving. This finding is consistent with emerging research on such curricula, for example, the recent work of O'Callaghan (1998) on Computer-Intensive Algebra. We also found that students who devote a great deal of time to practice of symbolmanipulation routines develop greater proficiency at those skills than students who spend much less time practicing symbolic calculation. The question facing those responsible for planning school mathematics curricula is what mathematics is most important for students to learn.

Our study does not provide information needed to answer the question about what mathematics is most worth learning, but it does suggest the kinds of trade-offs that might be expected when one allocates time to topics in ways that differ from allocations in the typical U.S. high school curriculum. Students in the Core-Plus Mathematics Project spend much less time on algebra topics than students would in a conventional high school program. Data from surveys of classroom-time allocation indicate that the major algebra units consumed about 180 class days of the 3 years of CPMP curriculum. Presumably, the additional time that would be applied to the study of algebra in a traditional curriculum is reallocated in the integrated CPMP curriculum to learning material from other content strands-especially statistics, probability, and discrete mathematics-that do not appear in traditional curricula. Data from other CPMP evaluations (Schoen \& Ziebarth, 1998) suggest that this effect does occur.

In the perspective on algebra taken in our study, we proposed that algebraic reasoning and problem solving involve three major component activities-formulation of algebraic representations for quantitative relationships, algebraic calculations like solving equations and inequalities to produce results beyond the given information, and interpretation of calculated results in the context of initial information and questions. The structure of our assessment instruments allowed us to compare curricular effects on each component of this process. Again, we found that if students are asked frequently to formulate mathematical models for situations and to interpret results of algebraic calculations, they develop greater understanding of and skill in those processes. With respect to the traditional processes of algebraic calculation, we found that although students in the CPMP program were not as proficient as control students at manipulation of symbolic expressions by hand, they had apparently learned a variety of alternative, calculator-based strategies for accomplishing the same goals. Furthermore, we found that students who
commonly did algebraic calculation in the context of meaningful problem settings developed some proficiency in using those situations as guides to their formal algebra. Whether that use of context cues is a strength or a disability was not clear from our study. Certainly, at some point educators want students to be able to deal with algebraic problems free of context cues. It might be that the reform curricula that commonly embed algebraic ideas in applied problem-solving explorations need to do a better job of helping students to abstract and articulate the underlying mathematical ideas. CPMP has attempted to make this improvement in its post-fieldtest version of the units.

The overall results suggesting achievement patterns related to curriculum content disguise very substantial differences in implementation and results at different sites. As Haimes (1996) discovered, teachers can use one of the new function-oriented approaches to algebra, with the best intentions to implement it as designed, but fail to realize the spirit of the new curriculum. There are very strong traditions in education that work against significant change in the content or the pedagogy of school mathematics (Gregg, 1995). We checked with both control and CPMP teachers to assess the extent to which our achievement testing was measuring effects of different curricula and teaching methods. Although we found general adherence to the intentions of the CPMP curriculum, we also found very significant variations in implementation. One could interpret the achievement-testing results as evidence of robustness in both control and CPMP curricula. However, there were some interesting exceptions to the overall pattern of findings.

The student-achievement data from individual Sites 3 and 4 are particularly interesting from this perspective. Site 3 was the only school in which control students outperformed CPMP students on all three parts of the student assessment. Unfortunately, our interviews with control and CPMP teachers did not reveal any obvious explanation for this contrary result. The control students were taken from Algebra II classes using a textbook that reflects influence of reform recommendations in the NCTM Standards. For instance, several control teachers indicated that their students had done some data modeling with graphing calculators. Informal reports from site visits by CPMP program evaluators had raised some concerns about the quality of implementation of the CPMP instructional model at the site. The CPMP field-test teachers reported success in retaining more students in 3 years of mathematics than in previous years when they had used traditional curricula. Thus, although we compared achievement of students with comparable Grade 8 test scores, efforts to encourage less able students might have led to an implementation of the CPMP curriculum that did not develop the full potential of the most able students. These are really only conjectures that need to be explored in further studies that are better able to document the implementation of the CPMP curriculum.

The situation at Site 4 is interesting for a different reason. That was the only site at which CPMP students matched the performance of control students on the measure of algebraic symbol manipulation without access to calculators. Again, there is no definitive explanation for this anomalous result. Some informal remarks
by teachers from the site suggested that the CPMP curriculum had been supplemented with materials that gave students more practice on traditional algebraic skills. This practice was to some extent a response to concerns in the school community about preparation of students for conventional, skill-oriented collegeadmission and placement examinations. Although field-test teachers generally reported serious attempts to implement the CPMP curriculum as intended, inclusion of additional practice material is the sort of curriculum enhancement that almost all teachers do routinely. As in the case of Site 3, careful linking of student achievement to fidelity of curriculum implementation will require further studies. What our research shows is a composite of effects from a variety of interpretations of the CPMP intentions.

## Representations and Learning

One of the central tenets of most Standards-based reforms in mathematics is the proposition that students who learn to view mathematical ideas and techniques from a variety of linked representational perspectives will have knowledge that is retained longer and applied more effectively than that acquired in only one, largely symbolic, mode. Conventional algebra curricula are beginning to take advantage of the representational features of graphing calculators. But in none of the curricula studied by control students in our study were the developers as deeply committed as in the Core-Plus Mathematics curriculum to making functions and their graphic, numeric, and symbolic representations the heart of school algebra. Our data show that this commitment leads to greater facility in solving problems that require use of graphic, numeric, and verbal information forms as well as ability to translate information from one representation to another. These results are consistent with research reported by O'Callaghan (1998), Garner (1998), and others.

Despite several specific efforts to get students to explain their work on the various constructed-response items of our algebra assessments, especially to indicate when and (briefly) how they had used graphing calculators, the student test papers did not include enough written work to let us draw conclusions about how students used calculators and alternative representations in their problem solving. Anecdotal reports from members of the research team who helped administer the assessments suggest that students (both CPMP and control) did, in fact, use calculators often. But, to get more dependable insight into that technology use, a more intensive clinical-interview study is needed.

## Conceptual and Procedural Knowledge

When the NCTM Curriculum and Evaluation Standards were released in 1989, they seemed to reflect an overwhelming national consensus on the direction that needed reforms should take. But as proposed Standards-based curricula emerge from development into broader use, there has been angry dissent from that consensus (Addington \& Roitman, 1996; Sowder, 1998). The heart of the controversy is almost always the balance between conceptual and procedural knowledge
in algebra. Proponents of change argue that students need not acquire as much symbolic-calculation skill as formerly; opponents of change argue that automaticity of such skills is essential to problem solving and further mathematical learning.

Our study does not settle this deep controversy. However, we believe that our results do provide support for the basic reform position. We found evidence that students with quite modest symbol-manipulation skills could outperform more symbolically capable students on tasks that required formulation of mathematical representations for problem situations and on tasks that required interpretation of calculated results. Furthermore, when those students had access to the kind of technological aids that are becoming standard mathematical tools, they could overcome limited personal calculation skills.

A similar conclusion can be drawn from the relatively weak correlations between student performance on conceptual and procedural problem-solving tasks. Those weak correlations suggest further that new approaches to algebra might well be enabling traditionally unsuccessful students to gain access to the problem-solving power of the subject. We did not find that students who were best at symbolic manipulations of algebra could also be expected to be best on conceptual tasks of mathematization and vice versa. When the symbol-manipulation powers of computer algebra systems become more widely available and user-friendly on inexpensive technologies, the significance of these findings will grow. All these findings concerning the interplay of conceptual and procedural knowledge are consistent with many findings in other aspects of mathematics (Hiebert, 1986; Hiebert \& Carpenter, 1992). However they run counter to the position of Anderson, Reder, and Simon (1996), whose research synthesis on the subject is often cited by critics of recent reform in school mathematics. This is clearly a complex subject that will require considerable additional study. Even then, the question that will remain is finding the right combination of conceptual and procedural knowledge in mathematics for students of different aptitudes and interests in the subject.

## Room for Growth

Whether proponent or opponent of Standards-based reforms, mathematics educators looking at the student-test scores reported in this study are likely to have the same disappointed reaction that we did in scoring the papers. There is clearly a great deal of room for improvement in the evident achievement of students in algebraic reasoning, problem solving, and calculation. Few students, CPMP or control, could do the kinds of basic symbolic calculation that is common fare on collegeadmission and placement tests. Even with access to powerful graphing calculators, many students could not accurately solve equations and inequalities. The general level of student performance on items requiring mathematization of applied quantitative problems was disappointingly low.

There are several plausible explanations for the generally low scores by CPMP and control students on all types of algebra items. Most reasonable is the fact that
for most students involved in the testing the results did not count at all toward their personal mathematics grades. Thus we might expect less than their best effort. Members of the research team who helped administer the testing did observe some students who were clearly not taking the assessment very seriously. However, there did not seem to be any evident bias in effort that could be expected to favor either CPMP or control groups, and our observations also suggested that quite a number of students gave serious attention to the various tasks.

A second possible explanation for the low student-test scores is that the assessments covered a broad spectrum of topics and skills in algebra and functions and, in the case of the CPMP students, they were not given immediately after completion of a major algebra unit. Thus we might well be seeing the normal decay of knowledge that occurs over time after students stop studying a topic intently.

When we looked at students' test papers, we were also disappointed at how poorly students explained their reasoning and strategies. Here again, the low personal stakes of the testing may have encouraged students to give minimal effort. However, another possible explanation is that typical testing practices in the United States do not encourage or require careful explanations of reasoning and that what we saw is what students expect to offer, even when test results matter to them personally.

Each of these perspectives on the low levels of student performance suggests some further studies to extend and clarify our findings. Smaller scale clinical-interview follow-up studies are likely to give a better reading on how students actually think about the tasks that were presented in paper-and-pencil format in this study. For instance, more complete data about students' calculator use would enhance our understanding of the effects of curricula like CPMP that make extensive use of such technology. One-on-one conversations with students would help us to see if their abilities to interpret algebraic calculations are as limited as their written work often suggests.

## Polishing the Stones

Our assessment of student algebra knowledge gained from the CPMP curriculum experience had a fundamental limitation caused by the nature of the curriculum development process. As we mentioned earlier, the participants (schools, teachers, and students) in our study were part of the CPMP national field test. The school administrators and teachers volunteered to participate in the study, so they might not be typical of those in U.S. high schools. The students tended to be somewhat below average in mathematical aptitude and prior achievement, as measured by standardized mathematics test scores from eighth grade. Most important, the students and their teachers were, each year, using a curriculum that was brand new and very different from typical high school mathematics. Perhaps the spirit of adventure from participation in this innovation experience inflated the effort and achievement of the CPMP classes. What seems more likely is that in subsequent years, when teachers become more polished in their delivery of the reform curriculum (and the curriculum materials themselves evolve in response to field-test experi-
ences), students should get a more focused and skillfully presented mathematics experience. Of course, if use of the CPMP program expands to schools and teachers with less thorough commitment, preparation, and support, it might be delivered with less polish and faithfulness to the fundamental principles on which it is based.

## Aspects of Algebra

The subject of algebra has a very long and broad history in mathematics, and one could argue that our study focused on only a few aspects of the subject, especially those that seem to be central to the CPMP view of appropriate algebra content in high school mathematics. True, our study did not probe student understanding of important abstract number-system structures that underlie procedures for symbol manipulation of polynomial and rational expressions. We did not ask questions that might give insight into the ways that students think about variables, expressions, equations, or functions. We did not pose tasks requiring the application of algebraic methods in geometry, statistics, probability, or discrete mathematics. Other studies should certainly describe and compare the understanding and skill that students in reform and traditional curricula develop in those areas. In all likelihood, the methodologies appropriate to such questions would require intensive study of a much smaller number of participants.

While admitting that there is more to knowing algebra than our study has assessed, we believe that the basic framework of issues that guided design and instrumentation for our study addresses central concerns. Few people would argue against the goal of helping students become effective in use of algebraic methods to describe and reason about relationships among quantitative variables. Although some reformers would argue against the centrality of symbol-manipulation skills in the emerging technological environment for mathematics, those skills are still prized by many mathematics educators, and they have been the heart of school algebra for most of the 20th century. We believe that we have investigated important issues, and we welcome contributions from others who see ways to investigate other aspects of algebra teaching and learning.

## CONCLUSIONS

The broad purpose of this study was to test the vision of reform proposals in recent advisory documents like the NCTM Standards by comparing effects of a curriculum designed to implement the Standards to those of more conventional curricula. We collected and analyzed extensive data on student learning of algebra from both kinds of curricula and found considerable support for main themes of the reform. However, no single study will provide complete or conclusive evidence.

In the discipline of mathematics, we are accustomed to finding logical arguments that conclusively affirm or deny propositions. In mathematics one counterexample falsifies a proposition, and no collection of positive illustrative examples firmly establishes truth. In mathematics education research, and in research outside
of mathematics more generally, the situation is very different. Even if one could consistently establish that Result B always follows from Practice A, there is the additional value question of whether Result B ought to be sought. Our study suggests some important patterns of consequences from curricular, instructional, and assessment practices in high school mathematics. Those patterns suggest areas in which both reform and traditional curricula need to be improved if they are to reach widely agreed-upon goals. But they also leave open the fundamental questions about what understanding and skill in algebra is most important for students to acquire from their school mathematics experience. Furthermore, they suggest some aspects of both reform and traditional curricula that need to be studied in more depth with methods other than those used in this study.

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## Authors

Mary Ann Huntley, Research Associate, National Center for Improving Science Education, 1726 M Street NW, Suite 704, Washington, DC 20036; Huntley@ NCISE.org
Chris L. Rasmussen, Assistant Professor, Department of Mathematics, Computer Science, and Statistics, Purdue University-Calumet, Hammond, IN 46323-2094; Raz@calumet.purdue.edu
Roberto Villarubi, Center for Mathematics Education, University of Maryland, College Park, MD 20742; rsv@math.umd.edu
Jaruwan Sangtong, Center for Mathematics Education, University of Maryland, College Park, MD 20742; Sangtong@wam.umd.edu
James Fey, Professor, Departments of Curriculum \& Instruction and Mathematics, University of Maryland, College Park, MD 20742; JF7@umail.umd.edu


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